PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Lunar Orbital Theory [and Discussion]

J. Kovalevsky and C. A. Murray

Phil. Trans. R. Soc. Lond. A 1977 **284**, 565-571
doi: 10.1098/rsta.1977.0032

Phil. Trans. R. Soc. Lond. A. **284**, 565–571 (1977) [565]
Printed in Great Britain

Lunar orbital theory

By J. Kovalevsky
Centre d'Etudes et de Recherches Géodynamiques et Astronomiques, 8 Boulevard Emile-Zola, 06130 Grasse, France

The present and expected accuracies of lunar laser ranging imply that the gravitational theory of the motion of the Moon should be consistent with at least the same precision. It is therefore necessary to aim at internal relative consistencies better than 10^{-11} or 10^{-12}.

Several theories based on numerical integration have been built and are currently being used in reducing the lunar laser ranging data. However, literal or semi-literal analytical theories have several important advantages over purely numerical ephemerides. This is why important programmes of building such theories are now in progress, particularly in the U.S.A. and in France.

Characteristics and the state of advancement of these theories will be reviewed and the possibility of constructing an analytical theory with the above mentioned accuracy discussed.

Introduction

The theory of the motion of the centre of mass of the Moon has always been, even in the early history of modern astronomy, a major challenge to mathematicians. From Isaac Newton to Ernest Brown, the accuracy of lunar theory has always been somewhat behind the actual precision of observations.

The last complete work published before the space age is the *Tables of the motion of the Moon* by Ernest Brown (1919). This theory is still used in astronomical ephemerides with some minor improvements introduced in 1960 and later (H.M. Nautical Almanac Office 1974). It is of some interest to analyse its accuracy.

Brown's theory includes a very accurate solution of the lunar main problem (the motion of the Moon disturbed by the Sun, assuming that the Earth moves on a keplerian ellipse around the Sun). This theory, together with the corrections introduced in his expressions by Eckert, Walker & Eckert (1966), has been shown by Henrard (1973) to have an accuracy of about 0.01″ in longitude and latitude and 0.0001″ in parallax. This represents a precision of about 20 m in all three coordinates, provided that the constants of integration are correct.

The perturbations due to the Earth flattening are less precise (Henrard 1973). A difference of 0.07″ can be found in longitude and latitude and probably corresponding inaccuracies exist in distances. But the worst part of the theory is the evaluation of planetary perturbations. Independently of biases due to incorrect constants of integration, a comparison of the improved lunar ephemeris with a numerical integration showed differences in position of the Moon of up to 500 m (Mulholland 1969). Inaccuracies in the values of fundamental or integration constants may increase this number, especially in considering the secular and long periodic terms. Mulholland (1972) quotes the probable presence of a 8″ per century secular error in longitude which brings the overall accuracy of this ephemeris up to more than 1 or even 2 km.

On the other hand, lunar-laser ranging gives distances between a point on the Earth and a point on the Moon to 10 cm and a limiting precision of 3 cm is expected to be reached in a few years. It is clear that such a precision is totally inconsistent with the accuracy of Brown's Lunar theory, even in its improved form. The observed minus computed quantities in the right-hand members of the equations of condition represent only the inaccuracy of the ephemerides, and cannot be used for the determination of other parameters present in the observed quantities. Actually, it is desirable that the internal consistency of the basic theory should be at least one order of magnitude better than the internal precision of the observations. This permits one to ignore the noise of the theory in comparison to the random noise of the observations. Of course, the various parameters entering in the theory cannot be known sufficiently well and have to be determined for the process of comparison of the observation to the theory. This can be achieved with such a precise ephemeris, provided that partial derivatives also exist for the same period of time.

Numerical integrations

Practically, this means that, for all times of observation – that is now about ten years – we need to have a consistent ephemeris with a precision of 1 cm, and shall soon need 1 mm. In relative precision, this is of the order of 10^{-11} to 10^{-12}. This is why, since the very beginning of the lunar-laser ranging experiment, the lunar ephemerides had to be so much better than those available in the Astronomical Ephemeris that something else had to be done. Numerical integration of the motion of the Moon, taking into account all the forces due to the non-spherical Earth, the Sun and all the major planets, with a sufficient number of significant figures, was the only possible method. Actually, it is the numerical solution of the general dynamical problem of the Solar System that is required, with an emphasis on the motions of the Moon.

This has been achieved by several teams in the U.S. Let me quote the work in Naval Weapons Laboratory using 40 000 optical observations of planets and the Moon made between 1911 and 1969 (Oesterwinter & Cohen 1972). In the J.P.L. the first ephemeris used for Lunar ranging was LE 16 (Garthwaite, Holdridge & Mulholland 1970), that included the positions of planets taken in another general integration program (DE 19, Devine 1967), while the source ephemeris for the Moon was based on Brown–Eckert improved theory. Later, this work was followed by LE 17 and then improved several times. Those specially designed for lunar laser are Lure 1 and Lure 2. Another series of integrations was performed in the University of Texas. Many of these ephemerides were not made available to the general scientific community. But Lure 2, which is based in particular on all laser observations covering the period August 1969 to June 1974, has been released at the end of 1974. It is believed (Calame 1975) that its external accuracy is of the order of a few tens of metres for a period of 10 years. But of course, the internal coherence is much better.

Many papers were published on various methods of numerical integration. We shall not discuss them here, and only refer to a recent review paper on this subject (Balmino 1975). Let us only stress the very great importance of a perfect documentation of all the parameters and assumptions included explicitly or implicitly in a numerical integration. This is not an easy task, much more difficult than in the case of analytical theories. Some of the difficulties were recently analysed by Mulholland (1975).

Analytical theories

So, numerical integrations, provided that some precautions are taken, can serve as the theoretical basis for the interpretation of the Lunar laser ranges. However, one needs not only the ephemeris, but also the partial derivatives with respect to all the parameters of the motion. These can be obtained numerically by performing several parallel integrations, as is done in the J.P.L. But if analytical expressions of these derivatives exist, it is much more efficient to use them. This has been done in the University of Texas and Joint Institute for Laboratory Astrophysics, and a detailed discussion of this method was made by O. Calame (1975). Building a new analytical theory sufficiently precise to become usable in the reduction of lunar distance observations has therefore become a major goal of celestial mechanics, as the modern aspect of the challenge quoted in the beginning of this presentation. It became possible to undertake such a work when sufficiently large and fast computers became available to permit the automatic treatment of large numerical or literal series that is needed.

The first attempt towards this direction is due to Barton (1966), who developed the disturbing function of the main lunar problem, and this was followed by various groups using techniques that were described by Kovalevsky (1968), Deprit (1968) and Rom (1971). Actually, two approaches to the lunar theory are possible and important efforts have been devoted to each of them: the literal theory and the semi-numerical theory. Since people have primarily studied the main problem, as the necessary step before computing other perturbations, we shall first examine this part of the theory, then conclude by some remarks on planetary and other perturbations.

Literal theories – main problem

In this approach, all parameters are kept in their purely literal form. The only assumption is that the metric parameters are small quantities and that it is possible to develop the coefficients in power series of these parameters. The convergence of these power series, as well as that of the general expressions, is not considered. What is really obtained is a truncated formal solution of the equations, and it is expected that when numerical values are given to parameters, the truncated expressions do represent the motion in a sufficiently long interval of time (Poincaré 1893).

All expressions, in the main problem, have the following form:

$$S = \Sigma R_{i_1 i_2 i_3 i_4}(m_0, e_0, e_0', \gamma_0, \alpha_0, \mu_0) \,{}^{\sin}_{\cos}\, (i_1 D + i_2 F + i_3 l + i_4 l'), \tag{1}$$

where the summation refers to four indices i.

D, F, l and l' are the four angular arguments of Delaunay. The expressions R are functions of the six small parameters. The four first are of the first order: m_0, ratio of mean motions, e_0, e_0', the mean eccentricities and $\gamma_0 = \sin \frac{1}{2}i_0$. The quantities $\alpha_0 = a_0/a'((E-M)/(E+M))$, where E and M are the masses of the Earth and of the Moon is of order 2. The last parameter,

$$\mu_0 = a_0/a'(EM/(E^2-M^2))$$

is of the fourth order.

The parameters actually used may be sometimes different to these, but for our purpose, this amounts to the same thing. The last work in this direction is due to Delaunay (1867), who obtained all terms to the seventh order. For a century, no attempt was made to resume this work.

A remarkably complete work was made in this subject by A. Deprit and his associates, J. Henrard & A. Rom, using the theory of Lie transforms (Deprit 1969). This work was described by Henrard (1973) and the series that are obtained are deposited at the U.S. Naval Observatory (Washington).

The expressions obtained for the longitude, latitude and parallax of the Moon have approximately 30 000 terms each. The developments in powers of small parameters are truncated in such a way that if ϕ is the global order in e_0, e_0' and γ_0 and ψ is the global order in m_0 and $\alpha_0^{\frac{1}{2}}$, one has

$$2\phi+\psi \leqslant 20$$

(in case of Delaunay, one had $\phi+\psi \leqslant 7$).

For numerical applications, these expressions were again truncated on numerical grounds and finally were kept, all terms having an amplitude of more than 0.00005″ in longitude and latitude, or 0.0000005″ in parallax.

This means that, in the present case, there remains a collection of quasi-random periodic terms of amplitudes less than 10 cm in the position of the Moon. This collection of random noise can certainly affect at least one, or possibly two orders of magnitudes more. Consequently, this theory, as such, does not encompass the specifications stated above for the use in Lunar ranging. At least two more significant figures are needed in order to be fully consistent within the centrimetre range required by the observations.

What is not clear, is the damage to the theory introduced by the numerical truncation. But anyhow, the most difficult problem is to be sure that no term larger than, say, a few millimetres has been disregarded. One has to study very carefully the apparent convergence of power series and of periodic terms before being confident in the theory to such an extent.

It is possible, in many cases, to improve the convergence of terms, using the Euler transform. With this artifice, Henrard (1973) has improved by a factor larger than 100 the main part of the secular motion of the perigee. However, this does not work for all terms. It seems, therefore, that some fundamental change in the presentation of terms may be necessary in order to go further in the precision of such a theory.

A similar literal solution of the main problem of the lunar theory is being constructed at the Bureau des Longitudes in Paris (Bec, Kovalevsky & Meyer 1973). It is based on a solution by successive approximations of equations derived from the Lagrange system with a different set of variables (Chapront & Mangeney 1969). Several approximations have been computed and an 8th-order theory is expected to be soon available. The procedures to improve the theory for very small terms will have to be completely different: it appears that the type of work necessary to obtain a large number of very small terms is quite different from the one that is necessary in order to build the main part of the solution. Divisors will have to be introduced, as well as a semi-numerical method to compute the small terms and assure the convergence in powers of m_0. In spite of the very large number of supplementary terms to be computed, the millimetre precision does not seem to be impossible for literal methods, provided that special procedures are adopted for those terms the derivatives of which do not need to appear in the expression of derivatives.

Semi-numerical theories – main problem

Another possible approach to Lunar theory is to follow the same procedure as Brown & Eckert and construct a semi-numerical theory. Here, the basic form of the solution is still (1), but the quantity R_{i_1, i_2, i_3, i_4} is no more expressed analytically in terms of the parameters, but is a number. This means that the parameters are given some *a priori* values kept all through the calculations and that they are not liable to be improved. In order that such theories may still represent a family of orbits, it is necessary to construct not only the coordinates of the Moon as Fourier series of D, F, l and l', but also the partial derivatives of the coordinates with respect to all the six parameters m_0, e_0, e'_0, γ_0, α_0 and μ_0 of the theory and eventually, if the parameters are not sufficiently close to the real ones, the second derivatives.

This task was undertaken in Bureau des Longitudes by M. Chapront-Touzé using again the same kind of method as in the case of the literal theory (Chapront & Mangeney 1969). The practical computing procedure, however, is very different and there is a difficulty that does not appear in the same way in literal theories, and that prevents the terms with small divisors to converge. This difficulty was overcome by writing the equations for the corrections to the coefficients of non-converging terms and neglecting the others (Chapront-Touzé 1974). This new method proved to be quite efficient in damping out the oscillations of the terms from one iteration to another.

In a first step, ten iterations were performed. They consisted of a simple integration of the equations obtained after the substitution in the right-hand members of the preceding iteration and then of the adjustment of the result to the constants of integration. At this stage, the oscillation of some terms became the dominant phenomena, so 8 other iterations were made, including the procedure for the stabilization of long-periodic terms.

The derivatives with respect to the parameters are computed by the same procedure, simply by writing that any coefficient A has the form

$$A = A_0 + \sum_{j=1}^{n} A_j \, \delta x_j,$$

where the δx_j are some literal correction to the n parameters, the squares of these quantities being neglected.

The final series for the principal part of this theory include terms in longitude and latitude of the order of 0.0001″. They are, as a rule, closer to the results of Deprit literal solution than to Brown–Eckert series. However, a couple of differences of the order of 0.01″ and some smaller remain between these theories.

This shows the great importance of having simultaneously several theories under formulation in order to be able to compare them to detect their inaccuracies or to discover the best procedures for further developments of the research in the field of the lunar theory.

Other perturbations

The success of the work on the main problem should not conceal the fact that the principal defect of Brown–Eckert theory lies in the evaluation of other perturbations, especially planetary perturbations.

In order to do this, it is necessary to start with a good solution of the main problem. This is the reason why this problem was first studied. Actually only preliminary steps have been taken towards a systematic study of these perturbations.

In the processs of building their analytical lunar theory, Deprit, Henrard & Rom (1971) have added to their solution of the main problem, the perturbations due to J_2, the main factor of Earth non-spherical gravitational field (Henrard 1973). Truncation errors can reach 0.005″ for some terms. It results that this part of the theory is not as accurate as the main problem.

Work on the problem of planetary perturbations has not yet – to my knowledge – resulted in actual expressions. Some partial derivatives of quantities depending on the Moon and necessary in the determination of planetary terms were computed by Griffith (1972). Deprit has worked also on this problem, but did not publish.

Actually, the problem is that one also needs a very complete theory of the motion of planets in order to get a good precision: as a matter of fact, a major reason of the inadequacy of Brown's expressions lies in the fact that the number of terms kept is much too small. It is also to be remarked that, in the theory of the motion of planets, some second order quasi-resonant terms are larger than many first order terms. This is why, before attempting to get any serious determination of the planetary perturbations, one should work on better analytical planetary theories.

Such a work is presently in progress in Bureau des Longitudes, under the leadership of J. Chapront. The analytical expressions of such theories are now in an advanced stage of computation of second and third order terms with respect to planetary masses (Simon & Bretagnon 1975; Chapront, Bretagnon & Mehl 1975). Methods of computation of negative powers of the distance between planets are now well established (Abu El Ata & Chapront 1975). It was shown that it is necessary to prepare the equations in such a way that variables referring to the Moon and those pertaining to the planets are separated until the integration process (Brown separation). This is done now, and the equations are written (Chapront & Abu El Ata 1976). The corresponding series are now being constructed as well from series of H. Chapront-Touzé as from those of A. Bec.

Although a lot remains to be done, it is expected that this coordinated effort should soon bring the first new results in the evaluation of planetary perturbations of the Moon.

Conclusions

Although serious efforts are being made to build new and more precise analytical theories of the motion of the Moon, there is still a great amount of work to be done before the challenge that lunar laser techniques have put before celestial mechanics is fully taken up. However, in recent years, serious progress has been made in the field of the main problem and the planetary perturbations are now also seriously considered by at least one group in the World.

But one may expect that many years will elapse before a complete analytical theory of the motion of the Moon will reach the precision required for use as reference ephemerides in lunar laser ranging. Until then, numerical integrations will remain the only possible method to get the necessary computational data for the analysis of these observations.

References (Kovalevsky)

Abu El Ata, N. & Chapront, J. 1975 *Astron. Astrophys.* **38**, 57.

Balmino, G. 1975 In *Satellite dynamics* (ed. G. E. O. Giacaglia), p. 50. Berlin: Springer-Verlag.

Barton, D. 1966 *Astron. J.* **71**, 438.

Bec, A., Kovalevsky, J. & Meyer, C. 1973 *The Moon*, vol. 8, p. 434.

Brown, E. W. 1919 *Tables of the motion of the Moon*. New Haven: Yale University Press.

Calame, O. 1975 Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris, 29 octobre 1975.

Chapront, J. & Abu El Ata, N. 1976 To be submitted to *Astron. Astrophys.*

Chapront, J., Bretagnon, P. & Mehl, M. 1975 *Celestial Mech.* **11**, 379.

Chapront, J., Chapront, M. & Simon, J. L. 1974 *Astron. Astrophys.* **31**, 155.

Chapront, J. & Mangeney, L. 1969 *Astron. Astrophys.* **2**, 425.

Chapront-Touzé, M. 1974 *Astron. Astrophys.* **36**, 5.

Delaunay, C. 1867 *Mem. Acad. Sci. Paris*, vol. **28** and **29**.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Deprit, A. 1968 *Astron. J.* **73**, 213.
Deprit, A. 1969 *Celestial Mech.* **1**, 12.
Deprit, A., Henrard, J. & Rom, A. 1971 *Astron. Astrophys.* **10**, 257.
Devine, C. J. 1967 *J.P.L. Tech. Rep.* no. 32–1181.
Eckert, W. J., Walker, M. J. & Eckert, D. 1966 *Astron. J.* **71**, 314.
Garthwaite, K., Holdridge, D. B. & Mulholland, J. D. 1970 *Astron. J.* **70**, 1133.
Griffith, J. S. 1972 *Celestial Mech.* **6**, 111.
Henrard, J. 1973 *Ciel et Terre*, **89**, 1.
H.M. Nautical Almanac Office 1974 *Explanatory supplement to the astronomical Ephemeris*, 3rd ed., p. 510. London: H.M. Stationery Office.
Kovalevsky, J. 1968 *Astron. J.* **73**, 203.
Mulholland, J. D. 1969 *Nature, Lond.* **223**, 247.
Mulholland, J. D. 1972 *Celestial Mech.* **6**, 242.
Mulholland, J. D. 1975 In *Satellite dynamics* (ed. G. E. O. Giacaglia), p. 127. Berlin: Springer-Verlag.
Oesterwinter, C. & Cohen, C. J. 1972 *Celest. Mech.* **5**, 317.
Poincaré, H. 1893 *Méthodes nouvelles de la mécanique céleste*, vol. 2, p. 1.
Rom, A. 1971 *Celestial Mech.* **3**, 331.
Simon, J. L. & Bretagnon, P. 1975 *Astron. Astrophys.* **42**, 259.

Discussion

C. A. Murray (*Royal Greenwich Observatory*). If we are to determine luni-solar precession from the observed motion of the Moon's node it is necessary that the theoretical motion be known with high accuracy. Can you set an upper limit to any uncertainties in this quantity, particularly those with long periods?

J. Kovalevsky. The errors in the theory of the motion of the Moon's node should certainly be one order of magnitude less than the accuracy required for the determination of precession.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES